Grading guide, Pricing Financial Assets, June 2016

1. Let the spot exchange rate for a foreign currency be S, denoting the value of one unit of the foreign currency as measured in the domestic currency.

   Assume that the exchange rate can be be modelled (under the original probability measure $\mathbb{P}$) by the geometric Brownian motion

   $$dS = \mu_S S dt + \sigma_S S dz$$

   where μ_S and $\sigma_S > 0$ are constants, and where dt and dz are the standard shorthand notations for a small time-step and a Brownian increment.

   (a) Describe the qualitative characteristics of this model, and discuss its possible shortcomings as a model of an exchange rate.

   (b) Assume that the domestic and foreign risk free interest rates are constants r and r_f, respectively. What will the drift rate of the exchange rate be under the domestic risk neutral probability measure ($\mathbb{Q}$) and a no-arbitrage assumption? Comment on the influence of the difference of domestic and foreign interest rates on the result.

   (c) Let $Z = 1/S$ be the exchange rate as measured in the foreign currency. Use Ito's lemma to find the process followed by Z where you eventually substitute S by $1/Z$.

Solution:

(a) The answer should cover an interpretation of the parameters of the model, and note the resulting continuous sample paths and the lognormal distribution of exchange rates or the normal distribution of returns. The student should note that realized sample paths may show large jumps (e.g. at a Central Bank change of policy). The student may also note that empirical return data tend to show changing volatility and excess kurtosis, but this is not directly covered by the syllabus (but in Hull chapter 19).

(b) See Hull section 16.5. The drift rate under $\mathbb{Q}$ will be $(r - r_f)$ (as we can treat the return on the foreign currency as we treat a continuously paid dividend on a stock).

(c) This is Hull Further Questions 16.27.

The transformation Z is independent of t, and you get

$$dZ = (-S^{-2}\mu_S S + 0.5 \cdot 2S^{-3}\sigma_S^2 S^2)dt + S^{-2}\sigma_S S dz$$

where we for the volatility use the sign convention that this is positive. By substitution we get

$$dZ = (\sigma_S^2 - \mu_S)Z dt + \sigma_S Z dz$$

(under $\mathbb{P}$) or equivalently

$$dZ = (\sigma_S^2 + r_f - r)Z dt + \sigma_S Z dz$$

(under $\mathbb{Q}$). A solution under either of the two measures is satisfactory. The drift of the exchange rate when measured in the foreign currency has the same volatility, but the drift changes, such that μ_S quite intuitively enters with a negative sign. However since the transformation is not linear, the two PDE are not symmetric, and a volatility adjustment also enters into the drift rate.

In what is not part of the syllabus, and thus not required of the answer, this asymmetry (Siegel's Paradox, Hull p. 676) is resolved if we change measure to the foreign risk neutral probability measure $\mathbb{Q}_f$ (where we should add $-\sigma_S^2$ to the drift, when changing measure).

2. In the Black-Scholes-Merton for options on a stock paying no dividends before the expiry of the option we get the following formula $c = S_0\Phi(d_1) - K\mathsf{e}^{-rT}\Phi(d_2)$ for the price c of a call at current time, where S_0 is the current stock price, K is the exercise price of the call, Φ is the cumulative standard normal probability function, r is a constant continuously compounded risk free interest rate, $d_1 = \frac{\ln\frac{S_0}{K}+(r+\frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}$, $d_2 = d_1 - \sigma\sqrt{T}$, and the stock price follows the Geometric Brownian Motion $dS = \mu S dt + \sigma S dz$, where z is a Brownian Motion.

   (a) In a model formulated by Merton the value of equity in a limited liability company with zero coupon debt of a notional of N maturing at T is considered as a call option on the value V of company assets. Assuming that the value of these assets can be described by a geometric Brownian motion derive a formula for pricing the zero coupon debt using the above model.

   (b) Comment on the model and its potential limitations.

   (c) Let $L = N\mathsf{e}^{-rT}/V_0$ and show that the credit spread on the zero coupon bond is

$$-\ln\left[\Phi(d_2) - \Phi(-d_1)/L\right]/T$$

Solution:

(a) Let the market value of equity be E_0 at time t. The market value of the debt will be $B_0 = V_0 - E_0$ assuming no costs or frictions in the form of third party payments in the event of default. Using the Black-Scholes-Merton formula to price the equity as a call on the assets with strike N and maturity T you get

$$\begin{aligned} B_0 &= V_0 - V_0\Phi(d_1) + N\mathsf{e}^{-rT}\Phi(d_2) \\ &= V_0\Phi(-d_1) + N\mathsf{e}^{-rT}\Phi(d_2) \end{aligned}$$

where we have altered the definition of d_1, d_2 to fit our interpretation, i.e.

$$d_1 = \frac{\ln\frac{V_0}{N} + (r + \frac{1}{2}\sigma_V^2)T}{\sigma_V\sqrt{T}}, d_2 = d_1 - \sigma_V\sqrt{T}$$

where σ_V is the volatility of the assets value.

(b) The student should discuss the assumptions made for the application of the BSM-model in general, and in particular for this application notice that the definition of a default is seldom as simply defined as just a failure to pay at the maturity of debt (e.g. it may be illegal for the management to continue ordinary business if a default is imminent, and incentives may also change the nature of the business - both such considerations makes it unlikely that σ_V is an exogenous constant).

(c) This is Hull Practice Question 23.23. Let s be the credit spread implicitly defined by

$$B_0 = \mathsf{e}^{-(r+s)T}N$$

By equating this with the formula above you get

$$\mathsf{e}^{-(r+s)T}N = V_0\Phi(-d_1) + N\mathsf{e}^{-rT}\Phi(d_2)$$

Substitute LV_0 for $N\mathsf{e}^{-rT}$ to get

$$LV_0\mathsf{e}^{-sT} = V_0\Phi(-d_1) + LV_0\Phi(d_2)$$

Divide by $LV_0 > 0$ to get

$$\mathsf{e}^{-sT} = \Phi(-d_1)/L + \Phi(d_2)$$

Taking the natural logarithm and dividing by $-T$ gives the desired result.

3. Assume that the stochastic instantaneous risk free rate is modelled by the one-factor

$$dr = m(r,t)dt + s(r,t)dz$$

where $s(r,t) > 0$, and dt and dz are the standard shorthand notations for a small time-step and a Brownian increment.

Consider two zero coupon bonds with different maturities T_i and values $V_i(r,t,T_i)$, $i \in \{1,2\}$, at $t < min[T_1, T_2]$.

(a) Construct a portfolio of a long position in 1 of the first bond and a short of Δ of the second. Denote the value of this portfolio Π and use Ito's lemma to find the delta that makes Π locally risk free.

(b) Find the drift rate of Π with this delta and use an arbitrage argument to characterise it.

Solution:

(a) This is an application of the development i Hull Section 27.1.

The portfolio will have the value $\Pi = V_1 - \Delta V_2$. By Ito's lemma we have that the stochastic element will be

$$\frac{\partial V_1}{\partial r} s(r,t) - \Delta \frac{\partial V_2}{\partial r} s(r,t)$$

Thus we can make the portfolio locally risk free by choosing

$$\Delta = \frac{\partial V_1}{\partial r} / \frac{\partial V_2}{\partial r}$$

with an intuitive interpretation.

(b) Using Ito's lemma the drift rate will be

$$\begin{aligned} &\tfrac{\partial V_1}{\partial r} m + \tfrac{\partial V_1}{\partial t} + 0.5 s^2 \tfrac{\partial^2 V_1}{\partial r^2} - \tfrac{\partial V_1}{\partial r} / \tfrac{\partial V_2}{\partial r} \left(\tfrac{\partial V_2}{\partial r} m + \tfrac{\partial V_2}{\partial t} + 0.5 s^2 \tfrac{\partial^2 V_2}{\partial r^2} \right) \\ =& \tfrac{\partial V_1}{\partial t} + 0.5 s^2 \tfrac{\partial^2 V_1}{\partial r^2} - \tfrac{\partial V_1}{\partial r} / \tfrac{\partial V_2}{\partial r} \left(\tfrac{\partial V_2}{\partial t} + 0.5 s^2 \tfrac{\partial^2 V_2}{\partial r^2} \right) \end{aligned}$$

and as it is locally risk free this must be equal to

$$r\Pi = r\left(V_1 - \left(\frac{\partial V_1}{\partial r} / \frac{\partial V_2}{\partial r}\right) V_2\right)$$

Equating these you get

$$\begin{aligned} &\left(\tfrac{\partial V_1}{\partial t} + 0.5 s^2 \tfrac{\partial^2 V_1}{\partial r^2} - r V_1\right) / \tfrac{\partial V_1}{\partial r} \\ =& \left(\tfrac{\partial V_2}{\partial t} + 0.5 s^2 \tfrac{\partial^2 V_2}{\partial r^2} - r V_2\right) / \tfrac{\partial V_2}{\partial r} \end{aligned}$$

We are thus left with one equation to determine V_1 and V_2.

The above will constitute a satisfactory answer.

However the student may also note that the LHS and the RHS depend on arbitrary T_1 and T_2. The equation can thus only hold if both are independent of the maturity.

Skipping the indexing referring to the maturity define

$$a(r,t) = \left(\frac{\partial V}{\partial t} + 0.5 s^2 \tfrac{\partial^2 V}{\partial r^2} - rV\right) / \frac{\partial V}{\partial r}$$

We can further define a risk premium λ implicitly by

$$a(r,t) = \lambda(r,t)s(r,t) - m(r,t)$$

Then we have a general bond pricing PDE

$$\frac{\partial V}{\partial t} + 0.5s^2 \tfrac{\partial^2 V}{\partial r^2} + (m - \lambda s)\frac{\partial V}{\partial r} - rV = 0$$

with a boundary condition $V(r,T,T) = 1$.